

# OPERATOR ALGEBRAS WITH CONTRACTIVE APPROXIMATE IDENTITIES II

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**ABSTRACT.** We make several contributions to our recent program investigating structural properties of algebras of operators on a Hilbert space. For example, we make substantial contributions to the noncommutative peak interpolation program begun by Hay and the first author, Hay and Neal. Another sample result: an operator algebra has a contractive approximate identity iff the linear span of the elements with positive real part is dense. We also extend the theory of compact projections to the most general case. Despite the title, our algebras are often allowed to have no approximate identity.

## 1. INTRODUCTION

For us, an operator algebra is a norm closed algebra of operators on a Hilbert space. In a long series of papers (e.g. [9, 5, 13, 12, 4, 27]), we and coauthors have investigated structural properties of approximately unital operator algebras in terms of certain one-sided ideals, hereditary subalgebras (HSA's), noncommutative topology (open and closed projections), etc. Most recently, we have proposed the set  $\frac{1}{2}\mathfrak{F}_A = \{x \in A : \|1 - 2x\|_{A^1} \leq 1\}$ , where  $A^1$  is the unitization of an operator algebra  $A$ , as the analogue of the positive part of the unit ball of a  $C^*$ -algebra, and this approach is proving fruitful [13, 11, 12]. In the present paper we advance this program in several ways. For example, in Sections 2 and 5 we tackle two important and related topics. First, we show that not all right ideals having a contractive approximate identity (cai) in a unital operator algebra are proximal (we recall that  $J$  is proximal in  $A$  if the distance  $d(x, J)$  is achieved for all  $x \in A$ ). Our example is quite simple, and solves a question that dates to the time of [8] (we recall the useful fact that two-sided ideals with cai, and more generally all  $M$ -ideals, are proximal [19]; as are closed one-sided ideals in  $C^*$ -algebras (see e.g. [15, 23]), and this is extremely important, as may be seen for example in the use made of proximality in the latter references). As a complement, we give a very natural sufficient condition that does ensure best approximation in such a right ideal. These allow us to make some substantial advances, and also to put some boundaries in place, in the noncommutative peak interpolation program begun by Hay in his Ph.D. thesis [20], and the first author, Hay and Neal [9, 21]. That is, we close in on the possible noncommutative generalizations of classical facts such as: if  $A$  is a function algebra on a compact space  $K$  and if  $E$  is *peak set* in  $K$  (defined below), then the functions in  $C(E)$  which are restrictions of functions in  $A$  to  $E$ , have norm preserving extensions in  $A$ . More generally, if  $f$  is a strictly

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*Date:* Revision of November 15, 2012.

2010 *Mathematics Subject Classification.* Primary 46L85, 46L52, 47L30, 46L07; Secondary 32T40, 46H10, 47L50, 47L55.

\*Blecher was partially supported by a grant from the National Science Foundation.

positive function on  $K$ , then the continuous functions on  $E$  which are restrictions of functions in  $A$ , and which are dominated by  $f$  on  $E$ , have extensions  $h$  in  $A$  satisfying  $|h| \leq f$  on all of  $K$  (see e.g. II.12.5 in [17]). In particular we give (in Sections 2, 5 and 6) several new noncommutative peak interpolation results which generalize such function algebra results. We remark that after this paper had been refereed we discovered one of our best noncommutative peak interpolation result, a generalization of the just mentioned classical result involving  $h$  and  $f$ , which we state here as Theorem 5.2. Moreover the new idea in the latter result yields short proofs [7] both of Read's theorem on approximate identities [27], and Hay's main theorem in [21, 20]. These are absolutely foundational results in the subject, and the extreme depth of their proofs hindered their accessibility until now.

In Section 3 we prove some results that should allow many facts from [13, 11, 12] to be generalized by widening the set  $\frac{1}{2}\mathfrak{F}_A$  to  $\mathfrak{r}_A = \{a \in A : \operatorname{Re}(a) = a + a^* \geq 0\}$ . This will have implications to our study of operator algebras in terms of a new kind of positivity, and we are investigating this in independent works. This does not mean that the  $\mathfrak{F}_A$  approach has been superseded; however using  $\mathfrak{r}_A$  instead will be adequate and simpler for certain applications. In Section 4 we present a mechanism to extend some of our results on approximately unital operator algebras to algebras with no kind of approximate identity. Indeed such an algebra has a HSA containing all other subalgebras having a contractive approximate identity (cai), and this HSA is the closure of the *linear* span of  $\frac{1}{2}\mathfrak{F}_A$ . Thus an operator algebra  $A$  has a cai iff the linear span of  $\mathfrak{F}_A$  or  $\mathfrak{r}_A$  is dense. In Section 6 we turn to noncommutative topology, and generalize the compact projections in the sense of the recent paper [12], and their theory, to algebras with no approximate identity. We also identify the right ideals which correspond to compact projections.

As with [13], many of our results apply immediately to give new results for function algebras, but we will not take the time to point these out.

We now turn to notation and more precise definitions. The reader is referred for example to [10, 9, 13] for more details on some of the topics below if needed. By an *ideal* of an operator algebra  $A$  we shall always mean a closed two-sided ideal in  $A$ . For us a *projection* is always an orthogonal projection. The letter  $H$  is reserved for Hilbert spaces. If  $E, F$  are sets, then  $EF$  denotes the closure of the span of products of the form  $xy$  for  $x \in E, y \in F$ . We often use silently the fact from basic analysis that  $X^{\perp\perp}$  is the weak\* closure in  $Y^{**}$  of a subspace  $X \subset Y$ . We recall that by a theorem due to Ralf Meyer, every operator algebra  $A$  has a unique unitization  $A^1$  (see [10, Section 2.1]). Below 1 always refers to the identity of  $A^1$  if  $A$  has no identity (we set  $A^1 = A$  if  $A$  has an identity of norm 1). If  $A$  is a nonunital operator algebra represented (completely) isometrically on a Hilbert space  $H$  then one may identify  $A^1$  with  $A + \mathbb{C}I_H$ . The second dual  $A^{**}$  is also an operator algebra with its (unique) Arens product, this is also the product inherited from the von Neumann algebra  $B^{**}$  if  $A$  is a subalgebra of a  $C^*$ -algebra  $B$ . The 'meet' or 'join' in  $B^{**}$  of projections in  $A^{**}$  remains in  $A^{**}$  (since for example if  $p, q$  are projections in  $A^{**}$  then it is well known that  $p \wedge q$  (resp.  $p \vee q$ ) is a weak\* limit of  $(pq)^n$  (resp. of polynomials in  $p$  and  $q$  with no constant term), which stays inside the von Neumann algebra  $\{x \in A^{\perp\perp} : x^* \in A^{\perp\perp}\} \subset B^{**}$ ). Note that  $A$  has a contractive approximate identity (cai) iff  $A^{**}$  has an identity  $1_{A^{**}}$  of norm 1, and then  $A^1$  is sometimes identified with  $A + \mathbb{C}1_{A^{**}}$ . Often our algebras and ideals are *approximately unital*, i.e. have a cai; when this is the case we will say so.

We recall that an *r-ideal* is a right ideal with a left cai, and an *ℓ-ideal* is a left ideal with a right cai. We say that an operator algebra  $D$  with cai, which is a subalgebra of another operator algebra  $A$ , is a HSA (hereditary subalgebra) in  $A$ , if  $DAD \subset D$ . See [9] for the theory of HSA's (a few more results may be found in [5, 13]). HSA's in  $A$  are in an order preserving, bijective correspondence with the r-ideals in  $A$ , and with the ℓ-ideals in  $A$ . Because of this symmetry we will usually restrict our results to the r-ideal case; the ℓ-ideal case will be analogous. There is also a bijective correspondence with the *open projections*  $p \in A^{**}$ , by which we mean that there is a net  $x_t \in A$  with  $x_t = px_t \rightarrow p$  weak\*, or equivalently with  $x_t = px_t p \rightarrow p$  weak\* (see [9, Theorem 2.4]). These are also the open projections  $p$  in the sense of Akemann [1] in  $B^{**}$ , where  $B$  is a  $C^*$ -algebra containing  $A$ , such that  $p \in A^{\perp\perp}$ . If  $A$  is approximately unital then the complement  $p^\perp = 1_{A^{**}} - p$  of an open projection for  $A$  is called a *closed projection* for  $A$ . We spell out some of the correspondences above: if  $D$  is a HSA in  $A$ , then  $DA$  (resp.  $AD$ ) is the matching r-ideal (resp. ℓ-ideal), and  $D = (DA)(AD) = (DA) \cap (AD)$ . The weak\* limit of a cai for  $D$ , or of a left cai for an r-ideal, is an open projection, and is called the *support projection*. Conversely, if  $p$  is an open projection in  $A^{**}$ , then  $pA^{**} \cap A$  and  $pA^{**}p \cap A$  is the matching r-ideal and HSA pair in  $A$ . It is well known that the closure  $J$  of a sum of r-ideals  $J_i$  is an r-ideal, but for the readers convenience we include a proof (using basic facts from e.g. [10, Section 2.5]). Obviously  $J$  is a right ideal, and  $J^{\perp\perp}$  is the weak\* closure  $K$  of the span of  $e_i A^{**}$ , where  $e_i$  is the support projection of  $J_i$ . If  $p = \vee_i e_i$ , then  $K \subset pA^{**}$  since  $e_i \leq p$ . Conversely,  $p \in J^{\perp\perp}$  by a remark above about meets and joins, and since  $J^{\perp\perp}$  is a weak\* closed right ideal we have  $pA^{**} \subset K$ . So  $J^{\perp\perp} = pA^{**}$  has left identity  $p$  and it follows from e.g. [10, Proposition 2.5.8] that  $J$  is an r-ideal (with support projection  $p$ ).

Let  $A$  be an operator algebra. If  $x \in A$  then  $\text{oa}(x)$  denotes the closed subalgebra of  $A$  generated by  $x$ . The set  $\mathfrak{F}_A = \{x \in A : \|1 - x\| \leq 1\}$  equals  $\{x \in A : \|1 - x\|_{A^1} = 1\}$  if  $A$  is nonunital, whereas if  $A$  is unital then  $\mathfrak{F}_A = 1 + \text{Ball}(A)$ . Many properties of  $\mathfrak{F}_A$  are developed in [13]. An  $x \in A$  is in  $\mathfrak{F}_A$  iff  $x + x^* \geq x^*x$ ; and  $x \in \mathbb{R}^+ \mathfrak{F}_A$  iff there is a constant  $C > 0$  with  $x + x^* \geq Cx^*x$ . If  $A$  is a closed subalgebra of an operator algebra  $B$  then it is easy to see, using the uniqueness of the unitization, that  $\mathfrak{F}_A = A \cap \mathfrak{F}_B$ . If  $x \in \frac{1}{2}\mathfrak{F}_A$  then  $\|x\| \leq 1$  and  $\|1 - x\| \leq 1$ . The following is a slight rephrasing of [13, Theorem 2.12], and it follows from its proof (we remark that there is a typo in the parallelogram law stated in that proof, the quantity after the '=' should be multiplied by 2):

**Lemma 1.1.** *Let  $T$  be an operator in  $B(H)$  with  $\|I - T\| \leq 1$ . Then  $T$  is not invertible if and only if  $\|I - \frac{1}{2}T\| = 1$ . Also,  $T$  is invertible iff  $T$  is invertible in the closed algebra generated by  $I$  and  $T$ , and iff  $\text{oa}(T)$  contains  $I$ . Here  $I = I_H$ .*

Generalizing Akemann's  $C^*$ -algebraic notion (see e.g. [2, 3]), a *compact* projection for an approximately unital operator algebra  $A$  is a closed projection in  $A^{**}$  with  $q = qa$  for some  $a \in \text{Ball}(A)$  (this may then be done with  $a \in \frac{1}{2}\mathfrak{F}_A$ ). See [12]. We recall that if  $A$  is a space of continuous functions on a compact set  $K$ , then a closed set  $E \subset K$  is called a *peak set* if there exists  $f \in A$  such that  $f|_E = 1$  and  $|f(x)| < 1$  for all  $x \notin E$ . These sets have been generalized to operator algebras in [20, 21, 9, 13, 12]. There are various equivalent definitions of peak projections in the latter papers (we warn the reader that if  $A$  is not unital then the definition late in [13, Section 2] is not equivalent and will not be considered here). If  $a \in \text{Ball}(A)$  define  $u(a)$  to be the weak\* limit of  $a(a^*a)^n$ . If this is a projection then this also

equals the weak\* limit  $\lim_n a^n$  (see [12, Lemma 3.1]), and in this case we will call  $u(a)$  a *peak projection* for  $A$  and say that  $a$  *peaks at*  $u(a)$ .

**Lemma 1.2.** *Let  $x \in \text{Ball}(B)$  for a  $C^*$ -algebra  $B$ , and let  $q$  be a closed projection in  $B^{**}$  such that  $xq = q$ . The following are equivalent:*

- (1)  $x$  peaks at  $q$  in the sense above (that is,  $q = u(x)$ ),
- (2)  $\|px\| < 1$  for every closed projection  $p$  in  $B^{**}$  with  $p \leq 1 - q$ ,
- (3)  $\|xp\| < 1$  for every closed projection  $p \leq 1 - q$ .

*Proof.* Item (2) is the same as (5) in [12, Lemma 3.1], but with the words ‘closed’ and ‘compact’ interchanged. However if one traces through the proof that (1) implies (5) there, one sees that one only really used that  $p$  is closed. Thus (1) is equivalent to (2). Since (1) is a symmetric condition, and  $xq = q$  iff  $qx = q$  since  $\|x\| \leq 1$ , we must have (1) equivalent to (3) too.  $\square$

**Lemma 1.3.** *For any operator algebra  $A$ , the peak projections for  $A$  are the weak\* limits of  $a^n$  for  $a \in \text{Ball}(A)$  in the cases that such limit exists.*

*Proof.* The one direction is proved in [12, Lemma 3.1], which is a slight generalization of Hay’s unital variant of that result [21]. Conversely, if the weak\* limit  $\lim_n a^n$  exists, then it is a closed projection  $q$ , indeed it is a peak projection, with respect to  $A^1$  by [9, Corollary 6.9]. By that result  $q = u(\frac{1+a}{2})$ , and by the last lines of the proof of [12, Theorem 3.4 (2)],  $q = u(x)$  for  $x = \frac{a+a^2}{2} \in \text{Ball}(A)$ .  $\square$

If  $a \in \frac{1}{2}\mathfrak{F}_A$  then the weak\* limit  $\lim_n a^n$  exists, and equals  $u(a)$  (see [12, Corollary 3.3]). If  $A$  is unital then  $u(a) = s(1 - a)^\perp$  and  $s(a) = u(1 - a)^\perp$  for  $a \in \frac{1}{2}\mathfrak{F}_A$ , where  $s(\cdot)$  is the *support projection* from [13, Section 2] (see [13, Proposition 2.22]). Compact projections for an approximately unital operator algebra are just the decreasing limits of peak projections [12, Theorem 3.4]. They are also the projections in  $A^{**}$  compact with respect to any  $C^*$ -algebra containing  $A$ , or with respect to  $A^1$ , by e.g. [12, Theorem 2.2]. If  $A$  is separable and unital (resp. approximately unital) then every closed (resp. compact) projection in  $A^{**}$  is a peak projection (see [13, p. 200] and [12, Theorem 3.4]). In Section 6 we generalize these facts.

## 2. PROXIMALITY AND NONCOMMUTATIVE PEAK INTERPOLATION

Peak interpolation focuses on constructing functions in a given algebra of functions on a topological space  $K$ , which have certain behaviors on fixed open or closed subsets  $E$  of  $K$ . Usually  $E$  (or its complement) is a peak set, or an intersection of peak sets. In [20, 21, 9] Hay, the first author, and Neal, began a program of noncommutative peak interpolation, with closed projections in  $A^{\perp\perp}$  playing the role of peak sets. A very nice application of these ideas appears in [29]. One may find several peak interpolation results in [20, 21, 9]. For example, our noncommutative Urysohn lemmas in [13, 12] are noncommutative peak interpolation theorems. Another example: in [21, Proposition 3.2] it is shown that if  $A$  is a unital operator algebra, if  $q$  is a closed projection in  $A^{\perp\perp}$ , and if we are given  $b \in A$  with  $\|bq\| \leq 1$ , and an  $\epsilon > 0$ , then there exists  $a \in (1 + \epsilon)\text{Ball}(A)$  such that  $aq = bq$ . In the commutative case one may take  $\epsilon = 0$  here, but it has been open for several years whether this is true for noncommutative algebras. In this section we answer this question. We had noticed earlier that the question was related to an open question that dates to the time of [8] about proximality (discussed in the introduction) which we settle too.

**Theorem 2.1.** *Suppose that  $A$  is an operator algebra (not necessarily approximately unital),  $p$  is an open projection in  $A^{**}$ , and  $b \in A$  with  $bp = pb$ . Then  $b$  achieves its distance to the  $\ell$ -ideal  $J = \{a \in A : ap = a\}$  associated with  $p$  (that is, there exists a point  $x \in J$  with  $\|b - x\| = d(b, J)$ ).*

*Proof.* It is well known, and straightforward, that for an  $\ell$ -ideal  $J$  with support projection  $p$  in a unital operator algebra  $A$ , if  $b \in A$  then  $d(b, J) = \|b(1 - p)\|$  (indeed in the second dual of  $A/J$ , which may be identified with  $A^{**}/J^{\perp\perp} = A^{**}/A^{**}p \cong A^{**}(1 - p)$ , the canonical copy of  $b + J$  corresponds to  $b(1 - p)$ ). Let  $D = pA^{**}p \cap A$ , the HSA associated with  $p$ . We have

$$bD = bpD = pbD \subset pA^{**}p \cap A = D,$$

and similarly  $Db \subset D$ . Now  $d(b, J) = \|b(1 - p)\|$  by the fact at the start of the proof applied in the unitization  $A^1$ . If  $C$  is the closed unital algebra generated by  $b$  and  $D$  and the identity of  $A^1$ , then  $D$  is an approximately unital ideal in  $C$ , and  $p$  is its support projection in  $C^{**} \cong C^{\perp\perp} \subset (A^1)^{**}$ . However approximately unital ideals are  $M$ -ideals (an observation of Effros and Ruan, see e.g. [10, Theorem 4.8.5 (1)]), and hence are proximal [19, 16]. Thus (and also applying the fact at the start of the proof again but now with respect to the ideal  $D$  of  $C$ ), there exists an element  $d \in D \subset J$  with  $\|b - d\| = d(b, D) = \|b(1 - p)\| = d(b, J)$ .  $\square$

Note that the last result applies to every  $\ell$ - or  $r$ -ideal in  $A$  (and to all  $b \in A$  commuting with the support projection of the ideal).

The following is an ‘ $\epsilon = 0$  variant’ of the peak interpolation result mentioned above Theorem 2.1. It is a generalization of a classical fact about peak sets [17] mentioned in our introduction.

**Corollary 2.2.** *Suppose that  $A$  is an operator algebra (not necessarily approximately unital),  $p$  is an open projection in  $A^{**}$ , and  $b \in A$  with  $bp = pb$  and  $\|b(1 - p)\| \leq 1$  (where  $1$  is the identity of the unitization of  $A$  if  $A$  is nonunital). Then there exists an element  $g \in \text{Ball}(A)$  with  $g(1 - p) = (1 - p)g = b(1 - p)$ .*

*Proof.* Suppose that  $x$  is a best approximation to  $b$  found in the previous proof in the  $\ell$ -ideal  $J$  supported by  $p$ , and let  $g = b - x$ . Then  $g(1 - p) = b(1 - p)$  since  $xp = px = x$  (the latter since  $x \in D$  in the proof of Theorem 2.1. If  $A$  is nonunital then  $J$  is also the  $\ell$ -ideal in  $A^1$  supported by  $p$ . Indeed  $A^{**}p \cap A = (A^1)^{**}p \cap A^1$  since if  $x \in A^1$  with  $x = xp \in A^{\perp\perp}$ , then  $x \in A^1 \cap A^{\perp\perp} = A$ . As stated in the previous proof, we have  $\|b(1 - p)\| = d(b, J)$ , which equals  $\|g\|$ . So  $g \in \text{Ball}(A)$ .  $\square$

It is of interest to replace the  $1 - p$  in the last result by a ‘compact’ projection  $q$  in  $A^{**}$ . This is possible, as we shall discuss in Section 5.

We now turn to showing that the results above are best possible.

**Theorem 2.3.** *Not every left ideal with a cai in a unital operator algebra is proximal.*

*Proof.* Identify  $c_0$  as the ‘main diagonal’ of  $\mathbb{K}(\ell^2)$ , and  $C$  as the ‘first column’ of  $\mathbb{K}(\ell^2)$ , and let  $A = C + c_0 + \mathbb{C}I$ , a closed unital subalgebra of  $B(\ell^2)$ . Explicitly,  $A$

consists of infinite matrices

$$a = \begin{pmatrix} v_1 & 0 & 0 & 0 & 0 & 0 & \dots \\ v_2 & d_2 & 0 & 0 & 0 & 0 & \dots \\ v_3 & 0 & d_3 & 0 & 0 & 0 & \dots \\ v_4 & 0 & 0 & d_4 & 0 & 0 & \dots \\ v_5 & 0 & 0 & 0 & d_5 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

with  $\vec{v} = (v_j) \in \ell^2$ , and  $(d_j) \in c$ , the space of all convergent sequences. Let  $J$  be the copy of  $c_0$  multiplied by  $I - E_{11}$ , which is an  $\ell$ -ideal in  $A$ , involving those matrices  $a$  as above with  $\vec{v} = \vec{0}$  and  $\lim_n d_n = 0$ . Set  $b \in A$  to be a matrix like  $a$  as above but with 1's on the main diagonal (so  $1 = v_1 = d_2 = d_3 = \dots$ ), and where  $v_j \neq 0$  for all  $j$ . Since  $JE_{11} = (0)$  and

$$\|\vec{v}\| = \|bE_{11}\| = \|(b - y)E_{11}\| \leq \|b - y\|, \quad y \in J,$$

we have  $d(b, J) \geq \|\vec{v}\| = \|\vec{v}\|_2$ . If  $e_n = E_{22} + \dots + E_{nn} \in J$  then one may write  $b - e_n$  as the sum of  $\vec{v}_n \oplus I$  and  $\vec{v} - \vec{v}_n$ , where  $\vec{v}_n = (E_{11} + e_n)\vec{v}$ . Hence

$$\|b - e_n\| \leq \max\{\|\vec{v}_n\|, 1\} + \|\vec{v} - \vec{v}_n\| = \|\vec{v}_n\|_2 + \|\vec{v} - \vec{v}_n\|_2 \rightarrow \|\vec{v}\|_2.$$

We deduce that  $d(b, J) = \|\vec{v}\|_2$ . We now show that this distance is not achieved. By way of contradiction suppose that  $y \in J$  with  $\|b - y\| = d(b, J)$ . Now  $b - y$  has first column  $\vec{v}$ , and for  $j > 1$  the  $j$ th column  $\vec{c}_j$  of  $b - y$  has at most one nonzero entry, and that entry is  $1 - y_j$ , where  $(y_j) \in c_0$ . If  $c$  is the matrix with two columns  $\vec{v}$  and  $\vec{c}_j$ , then

$$\|\vec{v}\| \leq \|c\| \leq \|b - y\|.$$

Hence if  $\|b - y\| = d(b, J) = \|\vec{v}\|_2$  then  $\|c^*c\| = \|c\|^2 = \|\vec{v}\|_2^2$ . However the first row of  $c^*c$  is  $(\|\vec{v}\|_2^2, \vec{v}_j(1 - y_j))$ , and so  $\vec{v}_j(1 - y_j)$  is forced to be 0, and hence  $y_j = 1$  for all  $j$ . But this is impossible since  $(y_j) \in c_0$ .  $\square$

**Corollary 2.4.** *Suppose that  $A$  is a unital operator algebra and that  $q$  is a peak projection for  $A$ . If  $b \in A$  with  $\|bq\| \leq 1$  then there need not exist an element  $g \in \text{Ball}(A)$  with  $gq = bq$ .*

*Proof.* Let  $A, J$ , and  $b$  be as in the proof of Theorem 2.3, with  $d(b, J)$  not achieved. Since  $A$  is separable, the complement of the support projection of  $J$  is a peak projection  $q$  by [13, Section 2] (see also [12, Theorem 3.4 (2)]). Let  $b' = b/\|bq\|$ . If there was an element  $g \in \text{Ball}(A)$  with  $gq = b'q$  then a slight modification to the proof of [9, Proposition 6.6] shows that  $d(b, J)$  is achieved.  $\square$

The last corollary shows that there is probably little point in looking for more sophisticated and completely general peak interpolation results of this flavor, other than the ones we have already found. Clearly the way to proceed from this point, in noncommutative peak interpolation, is to insist on a commutativity assumption of the type considered earlier in this section.

**Remark.** We can answer a question raised in [9, Section 6]. It follows from the above and [9, Proposition 6.6] that not every  $p$ -projection in the sense of [21, 9] is a ‘strict  $p$ -projection’ as defined above [9, Proposition 6.6].

Since  $r$ -ideals and  $\ell$ -ideals are examples of the (complete) one-sided  $M$ -ideals of [8] (see Proposition 6.4 there), we can also answer a question raised around the time of that investigation (see e.g. [14, Chapter 8]):

**Corollary 2.5.** *One-sided  $M$ -ideals in operator spaces in the sense of [8] need not be proximal.*

Note that  $A$  in our example above is very simple as an operator space, indeed it is a subalgebra of the nuclear locally reflexive  $C^*$ -algebra  $\mathbb{K}(\ell^2) + \mathbb{C}I$ . It is clearly exact and locally reflexive (since these properties are hereditary [26]). Thus we see that the obstacle to proximality, and to more sophisticated peak interpolation results than the ones we have already obtained, is not an operator space phenomenon, rather it is simply that one needs a certain amount of commutativity.

### 3. ELEMENTS WITH POSITIVE REAL PART

If  $A$  is any nonunital operator algebra then as we said earlier  $A^1$  is uniquely defined, and hence so is  $A^1 + (A^1)^*$  by e.g. 1.3.7 in [10]. We define  $A + A^*$  to be the obvious subspace of  $A^1 + (A^1)^*$ . This is well defined. To see this, suppose that  $A$  is a subalgebra of  $B(H)$ , and that  $\theta : A \rightarrow B(K)$  is a completely isometric (resp. completely contractive) homomorphism. By Meyer's result (see [10, Section 2.1], the map  $\lambda I_H + a \mapsto \lambda I_K + \theta(a)$  is completely isometric (resp. completely contractive), and by e.g. 1.3.7 in [10] it extends further to a unital completely isometric complete order isomorphism (resp. completely contractive unital) from  $\mathbb{C}I_H + A + A^* \rightarrow \mathbb{C}I_K + \theta(A) + \theta(A)^*$ , namely  $\lambda I_H + a + b^* \mapsto \lambda I_K + \theta(a) + \theta(b)^*$ . This last map restricts to a  $*$ -linear completely isometric (resp. completely contractive) surjection  $A + A^* \rightarrow \theta(A) + \theta(A)^* : a + b^* \mapsto \theta(a) + \theta(b)^*$ , for  $a, b \in A$ . Thus a statement such as  $a + b^* \geq 0$  makes sense whenever  $a, b \in A$ , and is independent of the particular  $H$  on which  $A$  is represented. We set  $\mathfrak{r}_A = \{a \in A : a + a^* \geq 0\}$ . This is a closed cone in  $A$ , and is weak\* closed if  $A$  is a dual operator algebra.

If  $x \in A$  with  $x + x^* \geq 0$  then  $x$  has a unique  $m$ th root  $x^{\frac{1}{m}}$  for each  $m \in \mathbb{N}$  with numerical range having argument in  $(-\frac{\pi}{m}, \frac{\pi}{m})$  (see [24, Theorem 0.1]). It is shown there that  $x^{\frac{1}{m}} \in \text{oa}(x)^1$ , but in fact obviously this root lies in  $\text{oa}(x)$  if the latter is nonunital (since if  $x^{\frac{1}{m}} = \lambda 1 + a$  for  $\lambda \in \mathbb{C}$  and  $a \in \text{oa}(x)$  then by taking  $m$ th powers  $x - \lambda^m 1 \in \text{oa}(x)$ ). We thank the referee for providing a proof of the next result, which may be known to experts on sectorial operators.

**Theorem 3.1.** *If  $x \in B(H)$  with  $x + x^* \geq 0$  then  $x^{\frac{1}{m}} x \rightarrow x$  and  $\|x^{\frac{1}{m}}\| \rightarrow 1$  as  $m \rightarrow \infty$ . Thus a normalization of  $(x^{\frac{1}{m}})$  is a cai for  $\text{oa}(x)$ .*

*Proof.* We use the machinery in [24, Theorem 1.2]. In particular, let  $\theta$  be a number slightly bigger than  $\frac{\pi}{2}$ , and let  $\Gamma$  be the positively oriented closed curve in the plane with three pieces: the two line segments  $\Gamma_1$  and  $\Gamma_3$  from 0 to the two points  $Re^{i\theta}$  and  $Re^{-i\theta}$ , and  $\Gamma_2$  the right part of the circle radius  $R$  centered at the origin connecting to the latter two points. We choose  $R$  so that the numerical range of  $x$  is inside the circle radius  $R$ . Define  $\Gamma_1(\epsilon)$  to be  $\Gamma_1$  with the last part of it, a segment of length  $\epsilon$ , removed. Similarly define  $\Gamma_3(\epsilon)$  to be the part of  $\Gamma_3$  bounded away from 0. Let  $\Gamma(\epsilon)$  be the part of  $\Gamma$  comprised by  $\Gamma_3(\epsilon)$ ,  $\Gamma_2$  and  $\Gamma_1(\epsilon)$ ; this is a connected but not closed curve, and [24] defines  $x^t = \lim_{\epsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_{\Gamma(\epsilon)} \lambda^t (\lambda 1 - x)^{-1} d\lambda$ , for  $t \in (0, 1)$ , showing that this limit exists in norm. Next consider the integral  $\frac{1}{2\pi i} \int_{\Gamma} (\lambda^{1+t} - \lambda)(\lambda 1 - x)^{-1} d\lambda$ . It is known that since  $x$  is 'accretive', we have that  $\lambda(\lambda 1 - x)^{-1}$  is bounded on  $(\Gamma_1 \cup \Gamma_3) \setminus \{0\}$  (see e.g. [18, Lemma C.7.2 (v)]), and by continuity it is also bounded on  $\Gamma_2$ . Thus there is a constant  $K$  with

$$(3.1) \quad \|\lambda(\lambda 1 - x)^{-1}\| \leq K, \quad \lambda \in \Gamma \setminus \{0\}.$$

It follows that

$$\left\| \int_{\Gamma_1(\epsilon)} \lambda^t (\lambda 1 - x)^{-1} d\lambda \right\| \leq K \int_{\Gamma_1(\epsilon)} |\lambda^{t-1}| ds = K \int_{\epsilon}^R r^{t-1} dr = K \left( \frac{R^t}{t} - \frac{\epsilon^t}{t} \right).$$

A similar bound holds for  $\Gamma_3(\epsilon)$ . On  $\Gamma_2$  the continuous function  $(\lambda 1 - x)^{-1}$  is bounded, and since  $|\lambda^t| = R^t \leq \max\{1, R\}$  here, we see that  $\int_{\Gamma_2} \lambda^t (\lambda 1 - x)^{-1} d\lambda$  is bounded independently of  $t$ . Letting  $\epsilon \rightarrow 0$  we deduce that  $\|x^t\| \leq \frac{C}{t}$  for some constant  $C > 0$ .

We now consider contour integrals over  $\Gamma$ . Since  $0 \in \Gamma$  this is not the usual Riesz functional calculus, but rather an extended version of it of the type considered e.g. in [18]. By the functional calculus for such contours, we have

$$x^{1+t} - x = \frac{1}{2\pi i} \int_{\Gamma} (\lambda^{1+t} - \lambda)(\lambda 1 - x)^{-1} d\lambda = \frac{1}{2\pi i} \int_{\Gamma} (\lambda^t - 1)\lambda(\lambda 1 - x)^{-1} d\lambda.$$

We are using here the fact that  $\left\| \int_{\Gamma \setminus \Gamma(\epsilon)} (\lambda^t - 1)\lambda(\lambda 1 - x)^{-1} d\lambda \right\|$  is dominated, by Equation (3.1), by a constant times the length of  $\Gamma \setminus \Gamma(\epsilon)$ , which goes to 0. Indeed by the same argument, there is a constant  $D$  with  $\|x^{1+t} - x\| \leq D \int_{\Gamma} |\lambda^t - 1| ds$ . By Lebesgues dominated convergence theorem we deduce that  $x^{\frac{1}{m}} x \rightarrow x$  as  $m \rightarrow \infty$ .

As we said earlier,  $x^{\frac{1}{m}}$  has numerical range having argument in  $(-\frac{\pi}{m}, \frac{\pi}{m})$  (see [24, Theorem 0.1]). Write  $x^{\frac{1}{m}} = a_m + ib_m$  for selfadjoint  $a_m, b_m$  with  $a_m \geq 0$ . Suppose that  $k < m$ . Expand  $x^{\frac{k}{m}} = (a_m + ib_m)^k$ , and use this to write  $a_m^k$  as  $x^{\frac{k}{m}}$  minus various products of powers of  $a_m$  and  $b_m$ . Applying the triangle inequality to the the latter, one obtains

$$\|a_m\|^k \leq \|x^{\frac{k}{m}}\| + \sum_{j=1}^k \binom{k}{j} \|b_m\|^j \|a_m\|^{k-j}.$$

It follows that

$$\frac{Cm}{k} \geq \|x^{\frac{k}{m}}\| \geq \|a_m\|^k - \sum_{j=1}^k \binom{k}{j} \|b_m\|^j \|a_m\|^{k-j} = \|a_m\|^k (2 - (1 + \frac{\|b_m\|}{\|a_m\|})^k).$$

By the numerical range fact above, there is a constant  $c > 0$  with

$$|\psi(b_m)| < \frac{c}{m} |\psi(a_m)| \leq \frac{c}{m} \|a_m\|,$$

for every state  $\psi$ . Thus  $\|b_m\| \leq \frac{c}{m} \|a_m\|$ . We deduce that  $\|a_m\|^k (2 - (1 + \frac{c}{m})^k) \leq \frac{Cm}{k}$ . If we ensure that  $k \leq \beta m$  for a fixed  $\beta$  with  $0 < \beta < \frac{\ln \frac{3}{2}}{c}$ , then for  $m$  and  $k$  large enough we have  $2 - (1 + \frac{c}{m})^k > \frac{1}{2}$ , and so  $\|a_m\| < (\frac{2Cm}{k})^{\frac{1}{k}}$ . If  $k$  dominates a small constant times  $m$ , it follows that  $\limsup_m \|a_m\| \leq 1$ . Hence  $\limsup_m \|x^{\frac{1}{m}}\| \leq 1$ . On the other hand, if a subsequence  $\|x^{\frac{1}{m_k}}\| \rightarrow \beta < 1$ , then this would contradict the fact that  $x^{\frac{1}{m}} x \rightarrow x$ . So  $\lim_m \|x^{\frac{1}{m}}\| \rightarrow 1$ .  $\square$

**Remark.** The fact that if  $x + x^* \geq 0$  then  $\text{oa}(x)$  has a cai was deduced in an earlier version of the paper from the next result and [13, Lemma 2.1].

**Theorem 3.2.** *If  $A$  is an operator algebra and  $x \in A$  with  $x + x^* \geq 0$  then  $\text{oa}(x) = \text{oa}(y)$  for some  $y \in \frac{1}{2}\mathfrak{F}_A$ . One may take  $y = x(I + x)^{-1}$ , and if so  $\|y\| \leq \frac{\|x\|}{\sqrt{1 + \|x\|^2}}$ .*



*Proof.* Suppose that  $A$  acts on a Hilbert space  $H$  and write  $I$  for  $I_H$ , and identify  $\text{oa}(x)^1$  with  $\text{oa}(I, x)$ , the operator algebra generated by  $\text{oa}(x)$  and  $I$  in  $B(H)$ . The numerical range of  $x$  in the latter algebra lies in the right hand half plane. We have  $-1 \notin \text{Sp}_{\text{oa}(I, x)}(x)$  and so  $I + x$  is invertible in  $\text{oa}(I, x)$ . Set  $y = x(I + x)^{-1} \in \text{oa}(x)$ . Then  $I - 2y = (I - x)(I + x)^{-1}$ , the Cayley transform of  $x$ , which is well known (see e.g. [28, IV, Section 4]), and is easy to see, is a contraction if  $x + x^* \geq 0$ . Hence  $y \in \frac{1}{2}\mathfrak{F}_A$ . Note that  $I - y = (I + x)^{-1}$ . It follows from Lemma 1.1 that  $\text{oa}(I - y)$  contains  $(I - y)^{-1} = I + x$  and  $I$ . So  $(I - y)^{-1} \in \text{oa}(I - y) \subset \text{oa}(I, y) \subset \text{oa}(I, I - y) = \text{oa}(I - y)$ . Hence  $x = y(I - y)^{-1} \in \text{oa}(y)$ . Thus  $\text{oa}(x) = \text{oa}(y)$ .

Representing  $A \subset B(H)$ , for  $\zeta \in H$  we have

$$\|(1 + x)\zeta\|^2 = \langle (1 + x + x^* + x^*x)\zeta, \zeta \rangle = \left(\frac{1}{\|x\|^2} + 1\right) \langle x^*x\zeta, \zeta \rangle = \left(1 + \frac{1}{\|x\|^2}\right) \|x\zeta\|^2.$$

Thus

$$\|x(I + x)^{-1}\zeta\| \leq \frac{\|x\|}{\sqrt{1 + \|x\|^2}} \|\zeta\| \leq \frac{\|x\|}{\sqrt{1 + \|x\|^2}},$$

for  $\zeta \in \text{Ball}(H)$ . So  $\|y\| \leq \frac{\|x\|}{\sqrt{1 + \|x\|^2}}$ .  $\square$

The following result was found by the first author, during a discussion with S. Sharma. The present proof however is an observation of the referee.

**Theorem 3.3.** *If  $A$  is any (not necessarily approximately unital) operator algebra then the closure of  $\mathbb{R}^+ \mathfrak{F}_A$  equals  $\{x \in A : x + x^* \geq 0\}$ .*

*Proof.* If  $x \in \mathfrak{F}_A$  then  $x + x^* \geq x^*x \geq 0$ . It follows that the closure of  $\mathbb{R}^+ \mathfrak{F}_A$  is contained in the closed cone  $\{x \in A : x + x^* \geq 0\}$ . For the converse, if  $x + x^* \geq 0$  then we appeal to the proof of the last result. We have  $x = \lim_{t \rightarrow 0^+} \frac{1}{t}tx(1 + tx)^{-1}$ , and  $tx(1 + tx)^{-1} \in \mathfrak{F}_A$ .  $\square$

We are currently working on implications of some of the results above with S. Sharma. We mention a few now. Following the ideas in the route presented in [13] one immediately obtains:

**Corollary 3.4.** (Cf. [13, Lemma 2.5]) *For any operator algebra  $A$ , if  $x \in A$  with  $x + x^* \geq 0$  and  $x \neq 0$ , then the left support projection of  $x$  equals the right support projection, and equals  $s(x(1 + x)^{-1})$ , where  $s(\cdot)$  is the support projection studied in [13]. If  $A \subset B(H)$  via a representation  $\pi$ , for a Hilbert space  $H$ , such that the unique weak\* continuous extension  $\tilde{\pi} : A^{**} \rightarrow B(H)$  is (completely) isometric, then this support projection  $s(x)$  also may be identified with the smallest projection  $p$  on  $H$  such that  $px = x$  (and  $xp = x$ ). That is,  $s(x)H = \overline{\text{Ran}(x)} = \text{Ker}(x)^\perp$ . Also,  $s(x)$  is an open projection in  $A^{**}$  in the sense of [9].*

We write  $s(x)$  for the support projection in the last result for  $x \in \mathfrak{r}_A$ .

**Corollary 3.5.** (Cf. [13, Corollary 2.6]) *For any operator algebra  $A$ , if  $x \in A$  with  $x + x^* \geq 0$  then the closure of  $x\overline{A}$  is an  $r$ -ideal in  $A$  and  $s(x)$  is the support projection of this  $r$ -ideal. We have  $\overline{x\overline{A}} = \overline{y\overline{A}} = s(x)A^{**} \cap A$ , where  $y \in \frac{1}{2}\mathfrak{F}_A$  is as in Theorem 3.2. The analogous results hold for  $\overline{Ax}$ , and this is the  $\ell$ -ideal matching  $x\overline{A}$ . Also,  $x\overline{Ax}$  is the HSA matching  $x\overline{A}$ .*

Thus our descriptions of  $r$ -ideals and  $\ell$ -ideals and HSA's from [13] in terms of  $\mathfrak{F}_A$ , may be rephrased in terms of the  $x \in A$  with  $x + x^* \geq 0$ . Corollaries 2.7 and 2.8 of [13] are true with  $x \in \mathfrak{F}_A$  replaced by  $\mathfrak{r}_A$ . Most of Lemma 2.10 of [13] also generalizes to this case, with the exception of (iv) and (v). Theorem 3.2 of [13] generalizes to this case too. We present the easy details of the proofs elsewhere.

Similarly, all results in [13, Section 8] generalize. For example, if one defines a map  $T : A \rightarrow B$  to be *real completely positive* if  $T(x) + T(x)^* \geq 0$  whenever  $x \in A$  with  $x + x^* \geq 0$  (and the obvious matching matricial version of this assertion holding for  $x \in M_n(A)$ , for all  $n \in \mathbb{N}$ ), then a map on an approximately unital operator algebra or operator system is real completely positive iff it is OCP in the sense of [13]. The following is the analogue of [13, Lemma 8.1].

**Corollary 3.6.** *Suppose that  $A$  is an approximately unital operator algebra. Then  $\{x \in A : x + x^* \geq 0\}$  is weak\* dense in  $\{x \in A^{**} : x + x^* \geq 0\}$ .*

*Proof.* We stated earlier that  $\{x \in A^{**} : x + x^* \geq 0\}$  is weak\* closed. If  $\eta$  is in the latter set, it is a limit of elements of the form  $t\eta$  for  $t \geq 0$  and  $\eta \in \mathfrak{F}_{A^{**}}$ , by Theorem 3.3. However  $t\eta$  is a weak\* limit of elements in  $t\mathfrak{F}_A \subset \{x \in A : x + x^* \geq 0\}$  by [13, Lemma 8.1].  $\square$

Many results from e.g. [11, 12] generalize too (i.e. using  $\{x \in A : x + x^* \geq 0\}$  in place of  $\mathfrak{F}_A$ ), by virtue of e.g. the rules for powers given in [11, Lemma 1.1], which are still valid by an obvious proof using [24, Corollary 1.3].

#### 4. NONUNITAL OPERATOR ALGEBRAS

If  $A$  is a unital or approximately unital operator algebra then  $\mathfrak{F}_A$  seems quite manageable. Hitherto we had assumed that  $\mathfrak{F}_A$  could be badly behaved if  $A$  was not approximately unital. However we shall see below that in this case  $\mathfrak{F}_A = \mathfrak{F}_C$  for an approximately unital subalgebra  $C$  (which might be  $(0)$ ).

If  $A$  is any operator algebra, define  $A_H = \overline{\mathfrak{F}_A A \mathfrak{F}_A}$ . This will play an important role in the sequel. We define  $A_r = \overline{\mathfrak{F}_A A}$  and  $A_\ell = \overline{A \mathfrak{F}_A}$ . By e.g. [13, Corollary 2.6] and the fact from the introduction that the closure of a sum of  $r$ -ideals is an  $r$ -ideal,  $A_r$  is an  $r$ -ideal. Similarly,  $A_\ell$  is an  $\ell$ -ideal. In fact these are the largest  $r$ -ideal and  $\ell$ -ideal in  $A$  (as may be seen using [13, Theorem 2.15]). By the proof in the introduction that the closure of a sum of  $r$ -ideals is an  $r$ -ideal, the support projection of  $A_r$  is  $p = \vee_{x \in \mathfrak{F}_A} s(x)$ , where  $s(x)$  denotes the support projection of  $x$  (see [13, Section 2]), since  $s(x)$  is the support projection of  $x\bar{A}$  (by e.g. Corollary 3.5). and joins). Similarly, the support projection of  $A_\ell$  is  $p$ , and now we see that  $A_\ell$  is the  $\ell$ -ideal associated with  $A_r$  (see [9, Section 2]). We also see that  $A_H = \overline{\mathfrak{F}_A A \mathfrak{F}_A} = A_r A_\ell$  is the HSA associated with this  $r$ -ideal (see [9, Section 2]).

**Proposition 4.1.** *An operator algebra  $A$  has a cai iff the span of  $\mathfrak{F}_A$  is dense in  $A$ .*

*Proof.* ( $\Rightarrow$ ) If  $A$  is unital then it is the span of  $\mathfrak{F}_A = 1 + \text{Ball}(A)$  obviously. Thus in the general case  $A^{**}$  is the span of  $\mathfrak{F}_{A^{**}}$ . If  $\varphi \in (\mathfrak{F}_A)^\perp$  then  $\varphi$  annihilates the span of the weak\* closure of  $\mathfrak{F}_A$ . This weak\* closure is  $\mathfrak{F}_{A^{**}}$  by [13, Lemma 8.1], and so  $\varphi = 0$  on  $A^{**}$  and so is zero. Thus  $\text{Span}(\mathfrak{F}_A)$  is dense.

( $\Leftarrow$ ) If the span of  $\mathfrak{F}_A$  is dense in  $A$  then  $A_r = A_\ell = A$ , using the existence of roots of elements of  $\mathfrak{F}_A$ . Hence  $A$  has a right cai and a left cai, and therefore has a cai (by e.g. [10, Proposition 2.5.8]).  $\square$

**Theorem 4.2.** *If  $A$  is any operator algebra then the closure of the linear span of  $\mathfrak{F}_A$  is a HSA in  $A$ . Indeed it is the biggest approximately unital operator algebra inside  $A$ , and equals  $A_H$ . Moreover  $\mathfrak{F}_A = \mathfrak{F}_{A_H}$ .*

*Proof.* Let  $D = \overline{\text{Span}}(\mathfrak{F}_A)$  and  $C = A_H$ . Clearly  $\mathfrak{F}_C \subset \mathfrak{F}_A$ . Conversely, since any  $x \in \mathfrak{F}_A$  has a third root, we have  $x \in C$ , so that  $x \in \mathfrak{F}_C$ . Thus  $\mathfrak{F}_C = \mathfrak{F}_A$ . Hence  $D = \overline{\text{Span}}(\mathfrak{F}_C) = C$  by Proposition 4.1, and it is clearly a HSA. If  $B$  is an approximately unital subalgebra of  $A$  then  $\mathfrak{F}_B \subset \mathfrak{F}_A = \mathfrak{F}_C$ , and so  $B \subset C$  by Proposition 4.1.  $\square$

**Corollary 4.3.** *Let  $A$  be any operator algebra.*

- (1)  $A_H = \overline{\text{Span}}(\mathfrak{r}_A)$ , and  $\mathfrak{r}_A = \mathfrak{r}_{A_H} \subset A_H$ .
- (2)  $A$  has a cai iff  $A = \overline{\text{Span}}(\mathfrak{r}_A)$ .

*Proof.* (1) The first assertion is obvious from Theorem 3.3 and Theorem 4.2. So  $\mathfrak{r}_A \subset A_H$ , and the second is now obvious.

(2) Follows from Theorem 3.3 and Proposition 4.1.  $\square$

Thus a finite dimensional operator algebra has an identity of norm 1 iff it is spanned by  $\mathfrak{r}_A$ , and it contains an orthogonal projection iff  $\mathfrak{r}_A \neq (0)$ . For the latter, note that if  $\mathfrak{r}_A \neq (0)$  then  $A_H$  is a nontrivial unital algebra, and so its identity is a projection.

We recall that  $A^2$  is the *closure* of the span of products of two elements from  $A$ .

**Corollary 4.4.** *If  $A$  is an operator algebra such that  $A^2$  has a cai, then  $A_r = A_\ell = A_H = \overline{\text{Span}}(\mathfrak{F}_A) = \overline{\text{Span}}(\mathfrak{r}_A) = A^2$ .*

*Proof.* Note that  $A_r$  and  $A_\ell$  are subsets of  $A^2$ , and  $(A^2)_r \subset A_r$ . If  $A^2$  has a cai then by the proof of Proposition 4.1 we have  $A^2 = (A^2)_r \subset A_r \subset A^2$ . So  $A_r = A^2$  and similarly  $A_\ell = A^2$ . The rest is clear from Theorem 4.2, Corollary 4.3 (1), and the definition of  $A_H$  (giving  $A^2 \subset A_H \subset A_r$ ).  $\square$

**Remarks.** 1) If  $A_r = A_\ell$  then this is the largest closed ideal with cai in  $A$ , since if  $J$  is any closed ideal with cai in  $A$  then  $J = J_r \subset A_r = A_\ell$ .

2) A similar result to Corollary 4.4 holds with  $A^2$  replaced by  $A^n$  for any  $n \in \mathbb{N}$ , with an almost identical proof.

One may use the above to extend much of the theory of operator algebras with cai, to arbitrary operator algebras. For example, there exist nice relationships between the states of a nonunital operator algebra  $A$  (defined as the nonzero functionals that extend to states on  $A^1$ ) and quasistates on  $A_H$  (we recall that a quasistate is a state multiplied by a scalar in  $[0, 1]$ ). If  $\varphi$  is a state on  $A^1$  then  $\varphi|_{A_H}$  is a quasistate on  $A_H$ . Indeed,  $\varphi$  extends further to a state on  $C^*(A^1)$ , and if  $p$  is the support projection of  $A_H$  then  $0 \leq \varphi(p) = \lim_t \varphi(e_t)$  for some cai for  $A_H$ , hence  $\varphi(p) \leq \|\varphi|_{A_H}\|$ . Conversely, by the Cauchy-Schwarz inequality, if  $x \in \text{Ball}(A_H)$  then  $|\varphi(x)| = |\varphi(xp)| \leq \varphi(p)$ , so that  $\varphi(p) \geq \|\varphi|_{A_H}\|$ . It follows by e.g. 2.1.18 and 2.1.19 in [10] that  $\varphi|_{A_H}$  is a quasistate on  $A_H$ . Note too that by [9, Theorem 2.10] a functional  $\psi$  on  $A_H$  has a *unique* Hahn-Banach extension  $\tilde{\psi}$  on  $A^1$ . The latter will be a state if  $\psi$  is, by the last line of 2.1.19 in [10] (taking  $A_H^1 = A_H + \mathbb{C}1_{A^1}$ , so that  $\tilde{\psi}(1) = 1 = \|\tilde{\psi}\|$ ).

From this it follows that results such as Lemma 2.9 in [13] are true for nonunital operator algebras too. However many results do not carry over. For example

$\mathfrak{F}_A$  need not be weak\* dense in  $\mathfrak{F}_{A^{**}}$  if  $A$  is nonunital (cf. [13, Lemma 8.1]). A counterexample is given by  $A$  equal to the functions in the disk algebra vanishing at 0. Here  $\mathfrak{F}_A = (0)$ , but  $\mathfrak{F}_{A^{**}} \neq (0)$  since  $A^{**}$  has many projections. Indeed for example the function  $f = z(z+1)/2 \in A$  peaks at 1, and by Lebesgue's theorem  $(f^n)$  converges weak\* to the canonical copy  $q$  in  $C(\mathbb{T})^{**}$  of the characteristic function of  $\{1\}$ , and  $q^2 = q$  and  $\|q\| \leq 1$ , so  $q$  is a nontrivial projection in  $A^{**}$ . That the bidual has many projections can also be seen since the bidual of its unitization, the disk algebra, does; and if  $A$  is any nonunital Arens regular Banach algebra, and  $A^1$  is a unitization of  $A$ , then as soon as  $(A^1)^{**}$  has nontrivial projections, then so does  $A^{**}$ . To see this consider the continuous homomorphism  $\chi : A^1 \rightarrow \mathbb{C} : a + c1 \mapsto c$ , whose canonical weak\* extension  $\tilde{\chi} : (A^1)^{**} \rightarrow \mathbb{C}$  has kernel  $A^{\perp\perp}$ . If  $p$  is a nontrivial projection in  $(A^1)^{**}$  then we have  $\tilde{\chi}(p) = 1$  or 0, and in the former case  $\tilde{\chi}(1-p) = 0$ . So either  $p$  or  $1-p$  is in  $\text{Ker}(\tilde{\chi}) = A^{\perp\perp}$ .

## 5. NONCOMMUTATIVE PEAK INTERPOLATION AGAIN

The following is a generalization of Theorem 2.1 and Corollary 2.2. The proof follows similar lines, but is a bit deeper.

**Theorem 5.1.** *Suppose that  $A$  is an operator algebra (not necessarily approximately unital), and that  $q$  is a closed projection in  $(A^1)^{**}$ . If  $b \in A$  with  $bq = qb$ , then  $b$  achieves its distance to the right ideal  $J = \{a \in A : qa = 0\}$ . If further  $\|bq\| \leq 1$ , then there exists an element  $g \in \text{Ball}(A)$  with  $gq = qg = bq$ .*

*Proof.* If  $\varphi \in A^\perp$ , then  $(qa)(\varphi) = 0$  for all  $a \in A$ , since  $q$  is weak\* approximable by elements in  $A^1$ , and  $A^1 A \subset A$ . Thus  $q$  satisfies the hypothesis of [21, Proposition 3.1], with  $X = A$ . It follows from that result that if  $J = (1-q)(A^1)^{**} \cap A$  then  $d(x, J) = \|qx\|$  for all  $x \in A$ .

Next, let  $\tilde{D} = (1-q)(A^1)^{**}(1-q) \cap A^1$ . By [9, Section 2],  $\tilde{D}$  is a HSA in  $A^1$ , and is approximately unital. Let  $C$  be the closed subalgebra of  $A^1$  generated by  $\tilde{D}$ ,  $b$ , and 1. Then  $\tilde{D}$  is an ideal in  $C$ : note for example that if  $d \in \tilde{D}$  then  $db = (1-q)db = d(1-q)b = db(1-q)$ , so  $db \in \tilde{D}$ . Since  $\tilde{D}^{\perp\perp} = (1-q)(A^1)^{**}(1-q)$ , by [9, Section 2], we have that  $1-q \in \tilde{D}^{\perp\perp} \subset C^{\perp\perp}$ , and so  $q \in C^{\perp\perp}$ . Indeed  $q$  is in the commutant of  $C$ , hence in the center of  $C^{\perp\perp}$ . Thus  $\tilde{D}^{\perp\perp} = (1-q)C^{\perp\perp}$ , and so  $\tilde{D} = (1-q)C^{**} \cap C$  is an  $M$ -ideal in  $C$ . The associated  $L$ -projection  $P$  onto the subspace  $\tilde{D}^\perp$  of  $C^*$ , is multiplication by  $q$ , since multiplication by  $1-q$  is the  $M$ -projection from  $C^{**}$  onto  $\tilde{D}^{\perp\perp}$ . Let  $I = C \cap A$ . It is an easy exercise that  $I$  is an ideal in  $C$ , for example using the fact that  $\tilde{D}A \subset A^1 A \subset A$  and similarly  $A\tilde{D} \subset A$ . Let  $D = I \cap \tilde{D} = \{x \in I : qx = 0\}$ .

If  $x \in I$  and  $\varphi \in I^\perp$  then  $q\varphi(x) = \lim_t \varphi(c_t x) = 0$  if  $(c_t)$  is a net in  $C$  with weak\* limit  $q$ , since  $c_t x \in I$ . We will make two deductions from this. First,  $P(I^\perp) \subset I^\perp$ . So by [19, Proposition I.1.16], we have that  $D = I \cap \tilde{D}$  is an  $M$ -ideal in  $I$ , hence it is proximal in  $I$ . Second, we deduce from [21, Proposition 3.1] with  $X = I$ , that  $d(x, D) = \|qx\|$  for all  $x \in I$ . So  $d(b, D) = \|qb\| = d(b, J)$ , the latter from the first paragraph of the proof. By proximality there exists a  $y \in D \subset J$  such that  $\|b - y\| = d(b, D) = \|qb\| = d(b, J)$ . We finish as before: setting  $g = b - y$  then  $gq = gq = qb$ , and so on.  $\square$

**Theorem 5.2.** *Suppose that  $A$  is an operator algebra (not necessarily approximately unital), a subalgebra of a unital  $C^*$ -algebra  $B$ . Identify  $A^1 = A + \mathbb{C}1_B$ .*

Suppose that  $q$  is a closed projection in  $(A^1)^{**}$ . If  $b \in A$  with  $bq = qb$ , and  $qb^*bq \leq qd$  for an invertible positive  $d \in B$  which commutes with  $q$ , then there exists an element  $g \in A$  with  $gq = qg = bq$ , and  $g^*g \leq d$ .

The proof of this is a small modification of the last proof [7]. We will however prove here a one-sided variant of Theorem 5.2.

**Lemma 5.3.** *Let  $A$  be a unital operator algebra, and suppose that  $q$  is a peak projection, the peak for an element  $a \in \text{Ball}(A)$ . Let  $B$  be a  $C^*$ -algebra generated by  $A$ . Suppose that  $b \in B$  and  $|b|$  commutes with  $a$ . Suppose also that  $qb^*bq \leq qd$ , for some invertible positive  $d \in B$  which commutes with  $a$ . Given an open projection  $u \geq q$  which commutes with  $a$ , and any  $\epsilon > 0$ , then there exists an  $n \in \mathbb{N}$  with  $\|bd^{-\frac{1}{2}}a^n\| \leq 1 + \epsilon$ , and  $\|bd^{-\frac{1}{2}}a^n(1 - u)\| < \epsilon$ .*

*Proof.* We work in  $B$ ; let  $e = 1_B$  and  $f = d^{\frac{1}{2}}$ . By Lemma 1.2 we have  $\|a(e - u)\| = r < 1$ , and so

$$\|a^n(e - u)\| = \|(a(e - u))^n\| \leq r^n \rightarrow 0.$$

So the last inequality in the Lemma will be easy. For the first, since  $a$  and hence  $q$  commutes with  $f, f^{-1}$ , and  $|b|$ , we have

$$f^{-1}qb^*bqf^{-1} = qf^{-1}b^*bf^{-1}q \leq f^{-1}qdqf^{-1} = q,$$

so that  $\|q|bf^{-1}|^2q\| \leq 1$ . Let  $b' = bf^{-1}$ , so that  $\||b'|q\| \leq 1$ , and let  $p$  be a spectral projection for  $[0, 1 + \epsilon)$  for  $|b'|$ . Then  $p$  is open, and it commutes with  $f^{-1}b^*bf^{-1} = |b'|^2$  and  $a^n$  (since  $a$  commutes with  $f^{-1}b^*bf^{-1} = |b'|^2$ ). Also  $pq = q$  since  $\||b'|q\| \leq 1$  (this is a nice exercise in the Borel functional calculus, or follows easily from [22, Corollary 5.6.31]). Then

$$\|bf^{-1}a^n\| = \||b'|a^n\| = \max\{\||b'|a^n(e - p)\|, \||b'|a^np\|\} \leq \max\{\||b'|\|a^n(e - p)\|, \||b'|p\|\},$$

which is dominated by  $1 + \epsilon$  for  $n$  large enough, since by the first lines of the present proof we can choose  $n$  with  $\|a^n(e - p)\| = r^n < (1 + \epsilon)/\||b'|\|$ .  $\square$

**Corollary 5.4.** *Suppose that  $A$  is an operator algebra (possibly not approximately unital), and that  $q$  is a projection in  $(A^1)^{**}$  such that  $q$  is the peak of an element  $a \in \text{Ball}(A^1)$ . Suppose that  $b \in A$ , such that  $|b|$  commutes with  $a$ . Let  $B$  be a unital  $C^*$ -algebra containing  $A$ , and suppose also that  $qb^*bq \leq qd$ , for some invertible positive  $d \in B$  which commutes with  $a$ . Then there exists an element  $x \in A$  with  $x^*x \leq d$  and  $xq = bq$ .*

*Proof.* We follow the classical idea due to Bishop (see II.12.5 in [17]), with some variations of our own. Set  $f = d^{\frac{1}{2}}$ . Letting  $\epsilon = \frac{1}{4}$ , by Lemma 5.3 with respect to  $A + \mathbb{C}1_B$ , there exists  $m_1 \in \mathbb{N}$  such that if  $g_1 = bf^{-1}a^{m_1}$  then  $\|g_1\| \leq \frac{5}{4}$ , and  $g_1q = bf^{-1}q = bqf^{-1}$ . Let  $u_n$  be the spectral projection for  $[0, 1 + \frac{1}{2^{n+1}})$  for  $|b'|$ , for  $n \geq 2$ , where  $b'$  is as in the last result. Note that  $(u_n)$  is a decreasing sequence of open projections, and  $u_n \geq q$  and  $u_n$  commutes with  $a$ , as in the proof of Lemma 5.3. By Lemma 5.3 again, we can choose an increasing sequence of positive integers  $(m_n)$  with  $\|bf^{-1}a^{m_n}(1 - u_n)\| \leq \frac{1}{2}$  and  $\|bf^{-1}a^{m_n}\| \leq 1 + \frac{1}{2^{n+1}}$ . Let  $g_k = bf^{-1}a^{m_k} = ba^{m_k}f^{-1}$ , and let  $g = \sum_{k=1}^{\infty} \frac{g_k}{2^k}$ , and  $x = gf = \sum_{k=1}^{\infty} \frac{g_k f}{2^k} \in A$ . We have  $gq = \sum_{k=1}^{\infty} \frac{g_k q}{2^k} = bf^{-1}q = bqf^{-1}$ , and so

$$xq = gfg = gqf = bq,$$

as desired. We also have

$$\|g(1 - u_2)\| \leq \frac{5}{8} + \sum_{k=2}^{\infty} \frac{\|g_k(1 - u_k)\|}{2^k} \leq \frac{5}{8} + \frac{1}{2} \sum_{k=2}^{\infty} \frac{1}{2^k} = \frac{7}{8} \leq 1.$$

If  $r$  is a projection dominated by  $u_n$  for every  $n$ , then

$$\|g_n r\| \leq \|g_n u_n\| \leq \| |b'| u_n \| \leq 1 + \frac{1}{2^{m+1}}, \quad m \in \mathbb{N},$$

so that  $\|g_n r\| \leq 1$ . Hence  $\|g r\| \leq 1$ . If  $s = u_{n-1} - u_n$  for some  $n \geq 3$ , then for  $k \geq n$  we have

$$\|g_k s\| \leq \|g_k(1 - u_n)\| \leq \|g_k(1 - u_k)\| \leq \frac{1}{2}.$$

If  $k < n$  then

$$\|g_k s\| \leq \| |b'| u_{n-1} \| \leq 1 + \frac{1}{2^n}.$$

Thus

$$\|g s\| \leq \sum_{k=1}^{n-1} \frac{\|g_k s\|}{2^k} + \sum_{k=n}^{\infty} \frac{1}{2^{k+1}} \leq (1 + \frac{1}{2^n})(1 - \frac{1}{2^{n-1}}) + \frac{1}{2^n} = 1 - \frac{1}{2^{2n-1}} \leq 1.$$

Since  $\|g\| = \| |b'| \sum_{k=1}^{\infty} \frac{a^{m_k}}{2^k} \|$ , and the  $(u_k)$  commute with  $|b'|$  and  $a^{m_k}$ , we see that  $\|g\|$  is dominated by the maximum of  $\|g(1 - u_2)\|$ ,  $\|g(\wedge_k u_k)\|$ , and  $\sup_{n \geq 3} \|g(u_{n-1} - u_n)\|$ , each of which is  $\leq 1$ . So  $g^*g \leq 1$  and  $x^*x = fg^*gf \leq f^2 = d$ .  $\square$

**Remarks.** 1) Corollary 5.4, Theorem 5.1, and the matching results in Section 2, lead to ‘Rudin-Carleson theorems’ of the type in [21, Proposition 3.4].

2) Corollary 5.4 is not true if one drops the condition that  $d$  commutes with  $a$ . For example, take  $A$  to be the lower triangular matrices in  $M_2$ ,  $q = a = E_{11}$ , and  $d = \epsilon E_{22} + |x\rangle\langle x|$ , where  $x = \begin{bmatrix} 1 & -1 \end{bmatrix}$ . We do not know if Corollary 5.4 is true if one replaces the condition that  $d$  commutes with  $a$  with  $d$  commuting with  $q$ .

## 6. COMPACT PROJECTIONS

In a recent paper [12] the first author and Neal developed (generalizing work of Akemann and coauthors, see e.g. [3]) the theory of compact projections in an approximately unital operator algebra  $A$ . We defined a projection  $q \in A^{**}$  to be *compact relative to  $A$*  if it is closed and  $qx = q$  for some  $x \in \text{Ball}(A)$ .

**Lemma 6.1.** *Let  $A$  be an approximately unital operator algebra. A closed projection  $q \in A^{**}$  is compact if  $qx = q$  for some  $x \in A$ .*

*Proof.* One direction is obvious. For the other, by [12, Theorem 2.2] we may assume that  $A$  is a  $C^*$ -algebra. We have  $x^*qx = q$ . There is a net  $(a_t)$  in  $A^1$  which decreases to  $q$  (since there is a net increasing to  $1 - q$ ), and if  $b = x^*a_t x$  then  $q \leq b \in A_+$ , which is one of the standard definitions of a compact projection (see e.g. the lines above [3, Lemma 2.7]).  $\square$

We thank C. Akemann for communicating to us the idea for Lemma 6.1 in the  $C^*$ -algebra case (it does not seem to appear in the literature).

Turning to the case of a (not necessarily approximately unital) operator algebra  $A$ , there are many possible notions of compactness which come to mind. Fortunately these collapse to two notions, one stronger than the other. We will simply say that a projection  $q \in A^{**}$  is *compact* (relative to  $A$ , or with respect to  $A$ ) if  $q$  is closed in

$(A^1)^{**}$  with respect to  $A^1$ . This is the same as  $q \in A^{\perp\perp}$  being compact with respect to  $B^{**}$  for a containing  $C^*$ -algebra  $B$ , by [12, Theorem 2.2] and the  $C^*$ -algebra case of that result.

**Theorem 6.2.** *Suppose that  $A$  is an operator algebra (not necessarily approximately unital), and that  $q \in A^{**}$  is a projection. The following are equivalent:*

- (1)  $q$  is compact with respect to  $A$  in the sense just defined.
- (2)  $q$  is closed with respect to  $A^1$  and there exists  $a \in \text{Ball}(A)$  with  $aq = qa = q$ .
- (3)  $q$  is a decreasing (weak\*) limit of projections of form  $u(a)$  for  $a \in \text{Ball}(A)$ .

*Proof.* (1)  $\Rightarrow$  (2) We have by e.g. [12, Theorem 3.4 (1)] that  $q$  is a decreasing limit of projections of form  $u(x)$  for  $x \in \text{Ball}(A^1)$ . Suppose that each  $x$  is of the form  $1 + a$  for some  $a \in A$ . Then  $x^n$  is of this form for each  $n$ , and if  $\chi : A^1 \rightarrow A^1/A$  is the canonical quotient then  $1 = \chi^{**}(x^n) \rightarrow \chi^{**}(u(x))$ , so that  $\chi^{**}(u(x)) = 1$ . Hence  $\chi^{**}(q) = 1$ , a contradiction. Thus at least one such  $x = a + c1$  for some  $a \in A, c \in \mathbb{C} \setminus \{1\}$ . Then

$$q = qu(x) = qu(x)x = qx = qa + cq,$$

so that  $qz = z = zq$  where  $z = \frac{1}{1-c}a$ . By Theorem 5.1 with  $d = 1$ , there exists  $a \in \text{Ball}(A)$  with  $aq = qa = q$ .

(2)  $\Rightarrow$  (3) Follows from the proof of [12, Theorem 3.4 (1)].

(3)  $\Rightarrow$  (1) Clear for example from [12, Theorem 3.4 (1)] since  $u(a)$  is a peak projection with respect to  $A^1$ .  $\square$

**Remark.** The condition in (3) is equivalent to  $q$  being an infimum of peak projections, as in [12, Theorem 3.4 (1)], and with the same proof (note that  $u(a) \wedge u(b) = u(\frac{a+b}{2})$  as in that paper, for  $a, b \in \text{Ball}(A)$  such that  $u(a)$  and  $u(b)$  are projections).

**Corollary 6.3.** *Let  $A$  be a (not necessarily approximately unital) operator algebra. If  $q \in A^{**}$  is compact then  $q$  is a weak\* limit of a net  $(a_t)$  in  $\text{Ball}(A)$  with  $a_tq = qa_t = q$  for all  $t$ .*

*Proof.* Choose  $a_t \in \text{Ball}(A)$  with  $u(a_t) \searrow q$  (see Theorem 6.2 (3)). Then  $qa_t^n = qu(a_t)a_t^n = qu(a_t) = q$ . Since the double weak\* limit  $\lim_t \lim_n a_t^n = \lim_t u(a_t) = q$ , a reindexing of  $(a_t^n)$  is a net of contractions  $y_t \rightarrow q$  weak\* with  $qy_t = y_tq = q$ .  $\square$

**Remark.** The fact in Corollary 6.3 was stated in [12] for approximately unital algebras before Theorem 2.1 there. Unfortunately there seems to be a typo there: the construction does not produce elements in  $\text{Ball}(A)$  in general. This is easily fixed though by choosing the  $e_t$  there in  $\frac{1}{2}\mathfrak{F}_A$  by Read's theorem [27, 7].

**Proposition 6.4.** *Suppose that  $A$  is a (not necessarily approximately unital) operator algebra.*

- (1) *If  $q$  is a projection in  $A^{**}$  and  $q = u(x)$ , for some  $x \in \text{Ball}(A^1)$ , then  $q = u(a)$ , for some  $a \in \text{Ball}(A)$ .*
- (2) *If  $A$  is separable then the compact projections in  $A^{**}$  are precisely the projections in  $A^{**}$  of the form  $u(a)$ , for some  $a \in \text{Ball}(A)$ .*

*Proof.* (1) Since  $u(x)$  is closed in  $A^1$ ,  $q$  is compact for  $A$ , and so by the previous result  $q = qb$  for some  $b \in \text{Ball}(A)$ . By the last lines of proof of [12, Theorem 3.4 (2)] we also have  $q = u(bx)$ .

(2) This follows from (1) and the fact from [12, Theorem 3.4 (2)] that since  $A^1$  is separable any projection compact with respect to  $A^1$  equals  $u(x)$ , for some  $x \in \text{Ball}(A^1)$ .  $\square$

We define a  $\mathfrak{F}$ -peak projection for  $A$  to be  $u(x)$ , the weak\* limit of the powers  $x^n$ , for some  $x \in \frac{1}{2}\mathfrak{F}_A$ . See [12, Corollary 3.3] for the fact that this weak\* limit exists and is a projection, which is nonzero if  $\|x\| = 1$ . We define a projection in  $A^{**}$  to be  $\mathfrak{F}$ -compact if it is a decreasing limit of  $\mathfrak{F}$ -peak projections. If  $A$  is approximately unital then the compact projections relative to  $A$  as defined above, the  $\mathfrak{F}$ -compact projections, and the compact projections in [12], are the same, by Theorems 2.2 and 3.4 in that reference. However if  $A$  is not approximately unital then there may exist compact projections in the sense above which are not  $\mathfrak{F}$ -compact projections. (This is the case in the example at the end of Section 4, where  $\mathfrak{F}_A = (0)$ , yet the copy of the characteristic function of  $\{1\}$  in  $C(\mathbb{T})^{**}$  equals  $u(f)$  for  $f = z(z+1)/2 \in A$ .)

**Proposition 6.5.** *If  $A$  is any operator algebra, then*

- (1) *A projection in  $A^{**}$  is  $\mathfrak{F}$ -compact in the sense above iff it is a compact projection in the sense of [12] for  $A_H$ .*
- (2) *A projection in  $A^{**}$  is a  $\mathfrak{F}$ -peak projection in the sense above iff it is a peak projection in the sense of [12] for  $A_H$ , and iff it equals  $u(a)$  for some  $a \in \text{Ball}(A_H)$ .*
- (3) *If  $A$  is separable then every  $\mathfrak{F}$ -compact projection in  $A^{**}$  is a  $\mathfrak{F}$ -peak projection.*

*Proof.* Most of these use the fact from Theorem 4.2 that  $\mathfrak{F}_A = \mathfrak{F}_{A_H}$ , together with facts stated in the introduction, or above the Proposition, about peak projections.

(2) By Theorem 3.4 in [12],  $q$  is a compact projection in the sense of [12] for  $A_H$ , iff it is a decreasing limit of terms of the form  $u(x)$  for  $x \in \frac{1}{2}\mathfrak{F}_{A_H} = \frac{1}{2}\mathfrak{F}_A$ . That is, iff it is  $\mathfrak{F}$ -compact.

(1) By definition  $q$  is  $\mathfrak{F}$ -peak iff  $q = u(x)$  for  $x \in \frac{1}{2}\mathfrak{F}_A = \frac{1}{2}\mathfrak{F}_{A_H}$ , which by what we said in the introduction is equivalent to the other conditions.

(3) This is obvious from (1), (2), and [12, Theorem 3.4 (2)], since in this case  $A_H$  is separable too.  $\square$

It now is a simple matter to generalize to algebras with no approximate identity other results from [12] concerning compact projections, using the two generalizations of compactness above. For the second,  $\mathfrak{F}$ -compactness, this is usually easier than for the first. We mention for example that results 2.3, 2.4, and 5.1 from that paper are clearly true for what we have called compact projections above, for algebras with no approximate identity (with all occurrences of the words ‘approximately unital’ removed). The proofs are unchanged. Also both noncommutative Urysohn lemmas from [12] do generalize:

**Theorem 6.6.** (Noncommutative Urysohn lemma for general operator algebras)  
*Let  $A$  be a (not necessarily approximately unital) operator algebra, a subalgebra of a  $C^*$ -algebra  $B$ , and let  $q$  be a compact projection in  $A^{**}$ . Then*

- (1) *For any open projection  $p \in B^{**}$  with  $p \geq q$ , and any  $\epsilon > 0$ , there exists an  $a \in \text{Ball}(A)$  with  $aq = q$  and  $\|a(1-p)\| < \epsilon$  and  $\|(1-p)a\| < \epsilon$ .*
- (2) *For any open projection  $p \in A^{**}$  with  $p \geq q$ , there exists  $a \in \text{Ball}(A)$  with  $q = qa, a = ap$ .*



*Proof.* (2) Apply [12, Theorem 2.6] in  $A^1$ : if  $a \in A^1, p \in A^{\perp\perp}$  and  $ap = a$ , then  $a \in A^{\perp\perp} \cap A^1 = A$  (since  $A^{\perp\perp}$  is an ideal in  $(A^1)^{**}$ ).

(1) The proof of [12, Theorem 2.1] works, taking  $(y_t)$  to be the net in 6.3.  $\square$

We say that a right ideal  $J$  in  $A$  is regular (resp. 1-regular) if there exists an  $x \in A$  (resp.  $x \in \text{Ball}(A)$ ) with  $(1 - x)A \subset J$ .

**Proposition 6.7.** *If  $q$  is a compact projection in  $A^{**}$  then the right ideal  $\{a \in A : qa = 0\}$  is 1-regular.*

*Proof.* If  $qa = 0$  with  $a \in \text{Ball}(A)$  then  $(1 - a)A \subset (1 - q)(A^1)^{**} \cap A$ .  $\square$

**Proposition 6.8.** *An  $r$ -ideal  $J$  in an approximately unital operator algebra  $A$  is regular iff it is 1-regular, and iff the complement of the support projection of  $J$  is a compact projection.*

*Proof.* Let  $p$  be the support projection of  $J$ , and suppose that  $x \in A$ . Then  $(1 - x)A \subset J$  iff  $p(1 - x)A = (1 - x)A$  iff  $p(1 - x) = (1 - x)$  iff  $p^\perp x = p^\perp$ . The result now follows from Lemma 6.1.  $\square$

**Remark.** 1) In the last proposition, one may further choose  $x \in \frac{1}{2}\mathfrak{F}_A$  if  $A$  is approximately unital [12].

2) Every  $r$ -ideal in a unital operator algebra  $A$  is 1-regular by [13, Theorem 1.2], but this is not true if  $A$  has a cai (this may be seen using Proposition 6.8 and the fact that some algebras have no compact projections [12]). We also remark that the first ‘iff’ in Proposition 6.8 is false with ‘approximately unital’ removed (as one may see in three dimensional algebras of upper triangular matrices).

**Corollary 6.9.** *Let  $A$  be a (not necessarily approximately unital) operator algebra. The following are equivalent:*

- (1) *There exist no nonzero compact projections in  $A^{**}$ ,*
- (2) *The spectral radius  $r(x) < \|x\|$  for all  $x \in A$ ,*
- (3) *The numerical radius  $\nu(x) < \|x\|$  for all  $x \in A$ ,*
- (4)  *$\|1 + x\| < 2$  for all  $x \in \text{Ball}(A)$ ,*
- (5)  *$(1 - x)A = A$  for all  $x \in \text{Ball}(A)$ .*

*Proof.* If any of these five conditions hold, then it is easy to see that  $A$  contains no projections. By [5, Theorem 3.5 and Proposition 3.7] we have that (2), (3), (4), (5) are all equivalent, and are also equivalent to every element of  $\text{Ball}(A)$  being quasi-invertible. However if  $q$  was a nonzero compact projection for  $A$ , then  $\{a \in A : qa = 0\}$  is not  $A$ , and so  $(1 - x)A \neq A$  by Proposition 6.7 and its proof, where  $q = qx$  with  $x \in \text{Ball}(A)$ .

Conversely, if  $\|1 + x\| < 2$  for  $x \in \text{Ball}(A)$ , then  $q = u(\frac{1+x}{2})$  is a nonzero projection by e.g. [12, Corollary 3.3], is in  $A^{**}$  by e.g. [9, Proposition 6.9(i)], and is closed in  $(A^1)^{**}$  like all peak projections.  $\square$

*Acknowledgements.* We thank Sonia Sharma for several discussions, and Damon Hay for reading a draft of the paper. We thank M. Neal for discussions on compactness of projections and the Urysohn lemma. We are deeply grateful to the referee for carefully reading the manuscript and making several helpful stylistic suggestions, and especially for providing, and allowing us to include, the beautiful proof of what is now Theorem 3.1. Also the current proof of Theorem 3.3 is due to the referee, and is shorter than our original one.

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